2.1 Involute of circle

Spiral is the fundamental geometry of the scroll. Shown in Figure 1 is the involute of a circle, which is the most commonly used spiral of scroll. The coordinates of this involute are dominated by the equations as Eq. (1) and Eq. (2).

$$x = a(\cos\phi + \phi\sin\phi) \tag{1}$$

$$y = a(\sin\phi - \phi\cos\phi) \tag{2}$$

where a is the basic circle radius, ϕ is the involute angle (corresponding to point M in the figure). The radius of the involute, ρ, at point M is then calculated by Eq. (3).

$$\rho = \sqrt{x^2 + y^2 - a^2} = \sqrt{a^2(1+\phi^2) - a^2} = a\phi \tag{3}$$

The differential length of the involute, dl, and the differential area, dS can be calculated by the following Eq. (4) and Eq. (5) respectively.

$$\mathrm{d}l = \rho\mathrm{d}\phi = a\phi\mathrm{d}\phi \tag{4}$$

$$\mathrm{d}S = \frac{1}{2}(a\phi)^2\,\mathrm{d}\phi \tag{5}$$

By integrating dl and dS from 0 to ϕ, the involute length, l, from the start point N to point M and the area between involute and basic circle, S, can be calculated by Eq. (6) and Eq. (7).

$$l = f_l(\phi) = \int_0^\phi a\phi\mathrm{d}\phi = \frac{1}{2}a\phi^2 \tag{6}$$

$$S = f_S(\phi) = \int_0^\phi \frac{1}{2}(a\phi)^2\,\mathrm{d}\phi = \frac{1}{6}a^2\phi^3 \tag{7}$$

2.2 Scroll geometry

Two identical scrolls are used to form the scroll expander, as shown in Figure 2. One of the scrolls is stationary and is called fixed scroll; while the other one is placed by rotating the fixed scroll by 180°, and is named as orbiting scroll since it is the one rotates. The inner and outer edges of the scroll are the involutes of the basic circle with the initial involute angles of α and $-\alpha$, respectively. The coordinates of these two involutes (x_i, y_i) and (x_o, y_o) are given by the equations as Eq. (8) to Eq. (11).

$$x_{\mathrm{i}} = a(\cos\phi_{\mathrm{i}} + (\phi_{\mathrm{i}} - \alpha)\sin\phi_{\mathrm{i}}) \tag{8}$$

$$y_{\mathrm{i}} = a(\sin\phi_{\mathrm{i}} - (\phi_{\mathrm{i}} - \alpha)\cos\phi_{\mathrm{i}}) \tag{9}$$

$$x_{\mathrm{o}} = a(\cos\phi_{\mathrm{o}} + (\phi_{\mathrm{o}} + \alpha)\sin\phi_{\mathrm{o}}) \tag{10}$$

$$y_{\mathrm{o}} = a(\sin\phi_{\mathrm{o}} - (\phi_{\mathrm{o}} + \alpha)\cos\phi_{\mathrm{o}}) \tag{11}$$

Points A to H in Figure 2 are illustrated to help the calculations of scroll parameters. Points A and B have the involute angle of π while points C and D are at 3π. Calculated by Eq. (8) to Eq. (11), the coordinates of points A to D are

A (-a, a(π-α)), B (-a, a(π+α)), C (-a, a(3π-α)), D (-a, a(3π+α))

Therefore the thickness of the scroll, t_s, and the pitch, *Pit* shown in the figure can be determined by the length of line segments AB (or CD) and AC (or BD), respectively by the following equations

$$t_{\mathrm{s}} = 2a\alpha \tag{12}$$

$$Pit = 2\pi a \tag{13}$$

Point G is one of the contact points of the orbiting scroll and the fixed scroll, which has the involute angle of 3π/2 for orbiting scroll (outer involute) and 5π/2 for fixed scroll (inner involute). To gain the perfect mesh, the orbiting radius, R_{or}, which has the equal value of the length of line segment EF, must satisfy the following equation

$$R_{\mathrm{or}} = l_{\mathrm{GE}} - l_{\mathrm{GF}} = a(\frac{5}{2}\pi - \alpha) - a(\frac{3}{2}\pi + \alpha) = \pi a - 2a\alpha \tag{14}$$

As seen from the figure, the end involute angle is $\phi_e = (2N_C + 1/2)\pi$, where N_C is the number of expansion chamber pairs. Hence the point H, which is the end point of outer involute of the fixed scroll, has the coordinate of $((\phi_e+\alpha)a, a)$. Considering the orbiting radius R_{or}, the radius of the shell shown in the figure is calculated by Eq. (15).

$$R_{\mathrm{sh}} = \sqrt{(a(\phi_{\mathrm{e}} + \alpha) + R_{\mathrm{or}})^2 + a^2} \tag{15}$$

2.3 Rotation of scroll

During the expander operation, the orbiting scroll rotates anticlockwise along the orbit with the radius of R_{or} without rotating around its own axis. Figure 3 exemplifies the operational cycle of the scroll expander with 3 pairs of expansion chambers (① to ③ shown in the figure) and one discharge chamber. The orbiting angle, θ, is defined as the angle between the horizontal axis and the connection line from the basic circle centre of the fixed scroll to the basic circle centre of the orbiting scroll. At the very beginning, $\theta = 0$ (or $\theta = 2\pi$), all the expansion chambers inside the expander reach their maximum volume after experiencing the previous rotation, while the discharge chamber has its minimum volume, as illustrated in Figure 3(a). When the orbiting scroll rotates, new chamber 1 is formed the volume of which starts from 0, while previous chamber 1 becomes new chamber 2, previous chamber 2 becomes new chamber 3, and previous chamber 3 merges with the discharge chamber. As the orbiting scroll keeps rotating, the expansion chambers 1-3 expand while the discharge chamber is subsequently compressed due to the constant total volume inside the expander shell, as shown in Figure 3(b), (c) and (d). Finally, all the chambers reach their initial states as $\theta = 2\pi$ (or $\theta = 0$).

Figure 4 is shown to help understanding the position of each mesh points between fixed and orbiting scrolls. At an orbiting angle θ, points K and L shown in the figure are the mesh points between chamber 1 and chamber 2, M and N are the mesh points between chamber 2 and 3 while O and P are the points separating chamber 3 and discharge chamber. The involute angle of point K at the fixed scroll, ϕ_K, equals to θ - $\pi/2$ as considering the perpendicular relationship between the tangent line of scroll involute at the mesh point and the connection line of O_1O_2. Thereafter other mesh points at the fixed scroll can be obtained by adding an interval of π. Meanwhile the involute angles of these mesh points at orbiting scroll are mirrored against those at fixed scroll, as summarized in Table 1, where F and F′ indicates points on the inner and outer involutes of the fixed scroll respectively, and O and O′ indicates points on the inner and outer involutes of the orbiting scroll respectively. More generally, the involute angles of mesh points can be determined by the following method, where i is the chamber number.

Fixed scroll: F′ $(-\pi/2 + \theta + 2(i-1)\pi)$ F $(\pi/2 + \theta + 2(i-1)\pi)$

Orbiting scroll: O $(\pi/2 + \theta + 2(i-1)\pi)$ O′ $(-\pi/2 + \theta + 2(i-1)\pi)$

2.4 Modification of the scroll

The geometry of the scroll needs to be modified to achieve high efficiency, light weight and easy manufacture. Figure 5 shows two typical modification methods. Figure 5(a) shows the scroll profile modified by a circular cutter, the intersection area of the cutter and the scroll is removed. The cutter centre is the cross point of the basic circle and the left side of x coordinate, hence the maximum radius of the cutter is $R_{mc} = a(\pi - \alpha)$ as shown in the figure. The cutter circle is governed by the following equation.

$$(x+a)^2 + y^2 = a^2(\pi-\alpha)^2 \tag{16}$$

The formed scroll has a sharp corner (point P in the figure), the involute angle of which can be calculated by substituting Eq. (10) and Eq. (11) for x and y respectively in the above equation, as expressed by Eq. (17) and further Eq. (18) after simplification.

$$(a(\cos\phi_P + (\phi_P+\alpha)\sin\phi_P)+a)^2 + (a(\sin\phi_P - (\phi_P+\alpha)\cos\phi_P))^2 = a^2(\pi-\alpha)^2 \tag{17}$$

$$(\phi_P+\alpha)^2 + 2\cos\phi_P + 2(\phi_P+\alpha)\sin\phi_P = (\pi-\alpha)^2 - 2 \tag{18}$$

Due to this modification, the volume of chamber 1 is not zero at $\theta = 0$ and the chamber is connected to chamber 2 through the emerged clearance caused by the missing of mesh points between the two chambers. The two chambers will start to be disconnected when point P becoming the mesh point. Thus, the disconnecting orbiting angle, θ_{disc}, can be determined by considering point P to be the mesh point F′ ($-\pi/2 + \theta + 2(i-1)\pi$) with $i = 1$. The following expression is then given.

$$\theta_{disc} = \phi_P + \frac{\pi}{2} \tag{19}$$

From a mechanical point of view, the sharp corner of the scroll is not favourable due to the high stress concentrations on this corner. Shown in Figure 5(b) is another modification by the so-called perfect mesh profile (PMP) [21-22] which consists of two circular arcs. The two arcs and the involutes are all tangential with each other at the contact points, so that the profile is smooth and continuous. To ensure the perfect mesh of the two scrolls, the two arcs must have the same unfold angle, λ, as shown in the figure. Point A with an involute angle ϕ_A at the outer involute of the scroll should be pre-selected as the modification start position. Then the end position of the modification is the point B at inner involution with the involute angle of ($\pi + \phi_A$). These two arcs have the diameter of R_1 and R_2 respectively, and the circle centre connection line O_1O_2 crosses the basic circle centre O. Because of the congruence of triangles ΔODO_1 and ΔOEO_2, $l_{OO_1} = l_{OO_2}$ and $l_{O_1D} = l_{O_2E}$, then the following expressions of R_1 and R_2 can be obtained.

$$R_2 - R_1 = R_2 - R_1 + l_{O_2E} - l_{O_1D} = l_{BE} - l_{AD} = \rho(B) - \rho(A) = R_{or} \tag{20}$$

$$l_{OC} = l_{OO_1} - R_1 = R_2 - l_{OO_2} = \frac{R_2 - R_1}{2} = \frac{R_{or}}{2} \tag{21}$$

$$R_1 = l_{OO_1} - l_{OC} = \frac{a}{\sin(\pi - \lambda)} - \frac{R_{or}}{2} = a(\frac{1}{\sin\lambda} + \alpha - \frac{\pi}{2}) \tag{22}$$

$$R_2 = R_1 + R_{or} = a(\frac{1}{\sin\lambda} - \alpha + \frac{\pi}{2}) \tag{23}$$

The angles including λ, β and γ in the figure must be calculated to complete the modification, which can be derived by the following steps

$$l_{O_1D} = \rho(A) - R_1 = a(\phi_A + \alpha) - a(\frac{1}{\sin\lambda} + \alpha - \frac{\pi}{2}) = a(\phi_A - \frac{1}{\sin\lambda} + \frac{\pi}{2}) \tag{24}$$

$$l_{O_1D} = a\cot(2\beta) \tag{25}$$

$$\cot(2\beta) = \phi_A - \frac{1}{\sin\lambda} + \frac{\pi}{2} = \phi_A - \frac{1}{\sin(\pi - 2\beta)} + \frac{\pi}{2} \tag{26}$$

$$\cot\beta = \phi_A + \frac{\pi}{2} \tag{27}$$

$$\lambda = \pi - 2\beta \tag{28}$$

$$\gamma = \phi_A - (\frac{\pi}{2} - (\pi - \lambda)) = \phi_A + \frac{\pi}{2} - \lambda \tag{29}$$

Similar to the modification by circular cutter, PMP modification results in the connection of the chamber 1 and 2 at the very beginning of the scroll rotation. The disconnection orbiting angle for such modification profile is equal to γ as the point C at the fixed scroll mesh with the same point of orbiting scroll.

2.4 Volume of chambers

Except chamber 1, each expansion chamber is enclosed by the inner involute of one scroll and outer involute of the other scroll between the two mesh points. Take the left side part of chamber 2 in Figure 4 as an example, the chamber is enclosed by the outer involute of fixed scroll and inner involute of orbiting scroll from mesh point K to M. Based on area calculation equation, Eq. (7), the enclosed area of this chamber 2 can be calculated by the following equation

$$S_{\mathrm{C2-left}} = f_{\mathrm{S}}(\mathrm{O(M)}) - f_{\mathrm{S}}(\mathrm{O(K)}) - [f_{\mathrm{S}}(\mathrm{F'(M)}) - f_{\mathrm{S}}(\mathrm{F'(K)})] = \frac{1}{6}a^2[(\phi_{\mathrm{M\text{-}o}} - \alpha)^3 - (\phi_{\mathrm{K\text{-}o}} - \alpha)^3] - \frac{1}{6}a^2[(\phi_{\mathrm{M\text{-}f}} + \alpha)^3 - (\phi_{\mathrm{K\text{-}f}} + \alpha)^3] \tag{30}$$

where the subscripts M-o, K-o, M-f, K-f indicate the point M and K at orbiting scroll and fixed scroll respectively. Using the involute angle given in Table 1, the area is calculated by the following equations

$$S_{\mathrm{C2-left}} = \frac{1}{6}a^2[\left(\frac{5\pi}{2} + \theta - \alpha\right)^3 - \left(\frac{\pi}{2} + \theta - \alpha\right)^3] - \frac{1}{6}a^2[\left(\frac{3\pi}{2} + \theta + \alpha\right)^3 - \left(-\frac{\pi}{2} + \theta + \alpha\right)^3] \tag{31}$$

$$S_{\mathrm{C2-left}} = 2\pi a^2(\pi + \theta)(\pi - 2\alpha) \tag{32}$$

Hence, the volume of the total chamber 2 is calculated by

$$V_{\mathrm{C2}} = 2h_{\mathrm{s}}S_{\mathrm{C2\text{-}left}} = 4\pi a^2 h(\pi + \theta)(\pi - 2\alpha) \tag{33}$$

where h_{s} is the scroll height. The more general area and volume calculation method ($2 < i < N_{\mathrm{C}}$) can be derived as

$$\begin{aligned} S_{\mathrm{C}i} &= 2[f_{\mathrm{S}}(\frac{\pi}{2} + \theta - \alpha + 2(i-1)\pi) - f_{\mathrm{S}}(\frac{\pi}{2} + \theta - \alpha + 2(i-2)\pi) \\ &\quad - f_{\mathrm{S}}(-\frac{\pi}{2} + \theta + \alpha + 2(i-1)\pi) + f_{\mathrm{S}}(-\frac{\pi}{2} + \theta + \alpha + 2(i-2)\pi)] \\ &= \frac{a^2}{3}[(\frac{\pi}{2} + \theta - \alpha + 2(i-1)\pi)^3 - (\frac{\pi}{2} + \theta - \alpha + 2(i-2)\pi)^3] \\ &\quad - \frac{a^2}{3}[(-\frac{\pi}{2} + \theta + \alpha + 2(i-1)\pi)^3 - (-\frac{\pi}{2} + \theta + \alpha + 2(i-2)\pi)^3] \\ &= 4\pi a^2(\pi - 2\alpha)(\theta - 3\pi + 2i\pi) \end{aligned} \tag{34}$$

$$V_{\mathrm{C}i} = h_{\mathrm{s}}S_{\mathrm{C}i} = 4\pi a^2 h_{\mathrm{s}}(\pi - 2\alpha)(\theta - 3\pi + 2i\pi) \tag{35}$$

The area and volume of chamber 1 can be determined by the geometric method with the aids of Figure 6 and Figure 7. As shown in Figure 6, the area of chamber 1 and parts of scroll at an arbitrary orbiting angle are divided by 12 small pieces of individual areas, then the area of chamber 1 can be expressed as

$$S_{\mathrm{C1}} = A_1 + A_4 + A_5 + A_7 + A_8 + A_{10} + A_{11} + A_{12} \tag{36}$$

Because of the symmetry, $A_1 = A_{12} + A_8$, $A_{11} = A_4$, $A_2 = A_6$, $A_5 = A_{10}$, then Eq. (36) is simplified as

$$S_{\mathrm{C1}} = 2\left(A_1 + A_4 + A_5\right) + A_7 \tag{37}$$

As shown in Figure 7(a), the area S_{11}, enclosed by basic circle and inner involute of fixed scroll between the two mesh points L and K, can be calculated by

$$\begin{aligned} S_{11} &= A_1 + A_3 + A_4 + A_5 + A_6 + A_8 + A_9 = f_{\mathrm{s}}(\mathrm{F(L)}) - f_{\mathrm{s}}(\mathrm{F(K')}) \\ &= \frac{a^2((\frac{\pi}{2}+\theta-\alpha)^3 - (-\frac{\pi}{2}+\theta-\alpha)^3)}{6} \end{aligned} \tag{38}$$

The area of the piece of scroll, S_{12}, shown in Figure 7(b) is

$$\begin{aligned} S_{12} &= A_3 + A_2 = A_3 + A_6 = f_{\mathrm{s}}(\mathrm{O'(L)}) - f_{\mathrm{s}}(\mathrm{O(L')}) \\ &= \frac{a^2((-\frac{\pi}{2}+\theta+\alpha)^3 - (-\frac{\pi}{2}+\theta-\alpha)^3)}{6} \end{aligned} \tag{39}$$

The area S_{13} which is the difference value between the shadow area and the yellow area shown in Figure 7(c) is

$$S_{13} = A_8 + A_9 - \frac{A_7}{2} = R_{\mathrm{or}}a - \frac{\pi a^2}{2} = \frac{a^2}{2}(\pi - 4\alpha) \tag{40}$$

Therefore the area S_{C1} and volume V_{C1} can be obtained as Eqs. (41) and (42).

$$S_{\mathrm{C1}} = 2(S_{11} - S_{12} - S_{13}) = \frac{a^2[(\frac{\pi}{2}+\theta-\alpha)^3 - (-\frac{\pi}{2}+\theta+\alpha)^3]}{3} - a^2(\pi - 4\alpha) \tag{41}$$

$$V_{\mathrm{C1}} = h_{\mathrm{s}} S_{\mathrm{C1}} \tag{42}$$

The volume of the discharge chamber ($i = N_{\mathrm{C}} + 1$) is obtained as

$$V_{\mathrm{dis}} = \pi h_{\mathrm{s}} R_{\mathrm{sh}}^2 - \sum_{i=1}^{i=N_{\mathrm{C}}} V_{\mathrm{C}i} - 2h_{\mathrm{s}} S_{\mathrm{scroll}} \tag{43}$$

where S_{scroll} is the projected area of scroll, which can be calculated based on Eq. (7) and has the following expression

$$S_{\mathrm{scroll}}=\frac{1}{6}a^2[(\phi_{\mathrm{e}}+\alpha)^3-(\phi_{\mathrm{e}}-\alpha)^3]=a^2\alpha(\phi_{\mathrm{e}}^2+\frac{\alpha^2}{3}) \tag{44}$$

However, the volume of chamber 1 and chamber 2 need to be re-calculated considering the scroll modification. The cut area by the circular cutter and the corresponding increased volume can be calculated by the process given in Appendix 1. Thereafter the volume of chamber 1 and 2 against the orbiting angle are

$$V_{\mathrm{C1\text{-}mc}}=V_{\mathrm{C2\text{-}mc}}=V_{\mathrm{C1}}+V_{\mathrm{C2}}+V_{\mathrm{mc}} \qquad \theta\leq\theta_{\mathrm{disc}} \tag{45}$$

$$\begin{cases} V_{\mathrm{C1\text{-}mc}}=V_{\mathrm{C1}}+V_{\mathrm{mc}} \\ V_{\mathrm{C2\text{-}mc}}=V_{\mathrm{C2}} \end{cases} \qquad \theta>\theta_{\mathrm{disc}} \tag{46}$$

The modification area and volume by using PMP method is given in Appendix 2, while the volume variations of chamber 1 and chamber 2 are more complicated and can be derived as given in Appendix 3. The following equations are obtained

$$V_{\mathrm{C1\text{-}mp}}=V_{\mathrm{C2\text{-}mp}}=V_{\mathrm{C1}}+V_{\mathrm{C2}}+V_{\mathrm{mp}} \qquad \theta\leq\gamma \tag{47}$$

$$\begin{cases} V_{\mathrm{C1\text{-}mp}}=h_{\mathrm{s}}(R_2^2\text{-}R_1^2)(\theta-\gamma-\sin(\theta-\gamma)) \\ V_{\mathrm{C2\text{-}mp}}=V_{\mathrm{C1}}+V_{\mathrm{C2}}+V_{\mathrm{mp}}-V_{\mathrm{C1\text{-}mp}} \end{cases} \qquad \gamma<\theta\leq\gamma+\lambda \tag{48}$$

$$\begin{cases} V_{\mathrm{C1\text{-}mp}}=V_{\mathrm{C1}}+V_{\mathrm{mp}} \\ V_{\mathrm{C2\text{-}mp}}=V_{\mathrm{C2}} \end{cases} \qquad \theta>\gamma+\lambda \tag{49}$$

3. Mathematic models of scroll expander – generator system

3.1 Valve model

Valves are installed before and after the expander to control the working fluid flow. The standard valve model is recommended by reference [23-24] as the following equation

$$\dot{m}_{\mathrm{valve}} \quad F_{\mathrm{P}}C_{\mathrm{v}}Y\sqrt{xP_{\mathrm{u}}\rho_{\mathrm{u}}} \tag{50}$$

where P_u and ρ_u are the upstream gas pressure and density, N_6 is the numerical constant and is 27.3 when pressure is in the unit of bar and mass flow rate is in the unit of kg hour^{-1}; F_P is the piping geometry factor which reflects the pressure loss due to the fittings attached directly to the inlet or outlet of the valve; C_v is the valve capacity coefficient, which is given by the valve manufacturer; Y is expansion factor accounting for the density change of the compressible fluid, which can be calculated by the following equation

$$Y = 1 - \frac{x_P}{3F_k x_T} \tag{51}$$

where x_P is the ratio of pressure drop, the following equation is given considering the chocked flow.

$$x_P = \min(\frac{P_u - P_d}{P_u}, F_k x_T) \tag{52}$$

$$F_k = \frac{k}{1.4} \tag{53}$$

where F_k is the ratio of specific heats factor, k is the expansion index (c_p/c_v), x_T is the critical pressure drop ratio factor which indicates the choked flow and is also given by the valve manufacturer.

3.2 Internal leakage area and leakage flow

The fixed scroll and orbiting scroll should mesh with each other perfectly during the rotation; however, the internal leakage inside the expander is inevitable in the reality. There are mainly two types of leakage between scroll chambers, flank leakage and radial leakage as illustrated in Figure 8(a) and (b), respectively. The flank leakage goes through the clearance between the side surfaces of two scrolls while the radial leakage goes through the clearance between one of the scroll tip wall and the base plate of the other scroll. The clearance size has been reported by several studies. Wang et al. [4-5] used constant values for both flank clearance δ_f and radial clearance δ_r at 0.01 mm and 0.015 mm respectively. Liu et al. [7-8] used 0.04 mm as both leakage size. Nevertheless, some researches [25-26] believed the clearances were varied as the changing pressure difference between chambers.

For the flank leakage, the length of the clearance is the height of the scroll, and because of the symmetry, the total length is

$$L_{\mathrm{f}} = 2h_{\mathrm{s}} \tag{54}$$

For the radial leakage, the clearance is in between the two mesh points of the scrolls, hence the length can be calculated based on Eq. (4) and has the equation related to the chamber number i and orbiting angle θ as given by Eq. (55).

$$L_{\mathrm{r}} = 2\int_{-\frac{\pi}{2}+\theta+2(i-1)\pi}^{\frac{\pi}{2}+\theta+2(i-1)\pi} a\phi \mathrm{d}\phi = 2\pi a\left(2\left(i-1\right)\pi+\theta\right) \tag{55}$$

The mass flow rate of the leakage flow can be referred to the orifice theory since that the leakage clearances are quite small comparing to the scroll thickness. The following equations can be used [5, 7, 8, 14]

[illegible] $$\sqrt{\frac{2k}{k-1}P_{\mathrm{i}}\rho_{\mathrm{i}}\left(\varepsilon^{\frac{2}{k}}-\varepsilon^{\frac{k+1}{k}}\right)} \tag{56}$$

[illegible] $$\sqrt{\frac{2k}{k-1}P_{\mathrm{i}}\rho_{\mathrm{i}}\left(\varepsilon^{\frac{2}{k}}-\varepsilon^{\frac{k+1}{k}}\right)} \tag{57}$$

where δ_{r} and δ_{f} are the leakage size, f_{r} and f_{f} are the flow coefficients of the working fluid, i is the chamber number, ε is the expansion ratio which is formulated by the following equation considering the chocked flow.

$$\varepsilon = \max\left(\frac{P_{i+1}}{P_i}, \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}\right) \tag{58}$$

Wang et al. [4] recommended the values of f_{r} and f_{f} at 0.87-0.95 for air as the working fluid.

3.3 Motion equation of the orbiting scroll

The pressure difference between each two adjacent chambers generate gas forces, including tangent force, radial force and axial force as shown in Figure 9(a) to (c).

The tangent force is the one which drives the rotation of the orbiting scroll. This force is perpendicular to the connection line of O_fO_o, thus the direction of the force is at an angle of $\theta + \pi/2$. Take the left side part of chamber 2 as an example, only part of the orbiting scroll (between mesh points N and M) involved in this chamber is imposed on the gas force generated by the pressure difference between chamber 2 and 3, while the other part of scroll (between mesh points K and N) faces the right side part of chamber 2. Meanwhile, in the right side part of chamber 2 only the fixed scroll is imposed on the gas force. Ignoring the thickness of the scroll, the tangent force on the orbiting scroll between chambers i and i+1 can be calculated by the following equation

$$\begin{aligned} F_{\mathrm{t}}(i) &= (P_i - P_{i+1})h_{\mathrm{s}}[2a(-\pi/2+\theta+2(i-1)\pi)+\pi a] \\ &= 2h_{\mathrm{s}}a[\theta+2(i-1)\pi](P_i - P_{i+1}) \end{aligned} \tag{59}$$

As shown in Figure 9(b), the radial force working on the orbiting scroll offsets the majority force by itself, only the part mapping the basic circle remains. Thus the following equation is used to calculate this force related to each chamber

$$F_{\mathrm{r}}(i) = 2h_{\mathrm{s}}a(P_i - P_{i+1}) \tag{60}$$

The axial force on the orbiting scroll is generated by the pressure difference between the two sides of the scroll plate as shown in Figure 9(c). The pressure can be calculated by

$$F_{\mathrm{a}} = \sum (P_i S_{\mathrm{C}i}) - P_{\mathrm{b}} S_{\text{orb-plate}} \tag{61}$$

where P_{b} is the pressure on the other side of orbiting plate.

The tangent force is the one to drive the rotation of orbiting scroll. As aforementioned, every point on the orbiting scroll rotates along a circle with the same diameter of R_{or} but different circle centre position. Hence the torque produced by the tangent force on the orbiting scroll is

$$T_{\mathrm{t}} = R_{\mathrm{or}} \sum_{i=1}^{N_{\mathrm{C}}} F_{\mathrm{t}}(i) \tag{62}$$

This torque is the drive moment of the rotation, and is used to conquer the mechanical friction torque (drag torque) T_{fr} and to provide the electromechanical torque T_{em} when a generator is connected. The equation describing the rotation of the expander - generator can be obtained as [27-28]

$$(J_{\mathrm{orb}} + J_{\mathrm{sha}} + J_{\mathrm{arm}} + J_{\mathrm{Old}}\sin^2\theta)\frac{\mathrm{d}\omega}{dt} - J_{\mathrm{Old}}\sin\theta\cos\theta\omega^2 = T_{\mathrm{t}} - T_{\mathrm{fr}} - T_{\mathrm{em}} \tag{63}$$

where J is the moment of the inertia of the rotating parts, ω is the angular velocity, the subscripts orb, sha, arm and Old indicate the orbiting scroll, the shaft, the generator armature and the Oldham ring, respectively. The calculation equations for these inertias are given as follows

$$J_{\mathrm{orb}} = m_{\mathrm{orb}} R_{\mathrm{or}}^2, J_{\mathrm{sha}} = m_{\mathrm{sha}} R_{\mathrm{sha}}^2 / 2,\ J_{\mathrm{arm}} = m_{\mathrm{arm}} (R_{\mathrm{arm\text{-}out}}^2 + R_{\mathrm{arm\text{-}in}}^2) / 2, J_{\mathrm{Old}} = m_{\mathrm{Old}} R_{\mathrm{or}}^2 \tag{64}$$

Because the Oldham ring performs reciprocating motion along a straight line but not the orbiting circle, its inertia should be modified by considering the rotating inertia along the orbiting circle and the orbiting angle as given in Eq. (63).

The mechanical friction torques are schematically illustrated in Figure 10. Thrust bearings exist on both surfaces of the orbiting scroll plate. The upper surface receives the thrust load from the cover of fixed scroll while the lower surface is enduring the friction with the surface of the supporting frame. The other major torques are on the journal bearings which support the crank shaft and the orbiting scroll. Moreover, there is also friction loss between the Oldham ring and its slots. These mechanical friction torques can be calculated by solving the force balance equations as reported in [27, 29-30], while some tests on friction loss and friction factors have been reported [31-33]. On the other hand, some studies suggested using constant mechanical loss [15, 34] or linearly varying mechanical loss with rotational speed [11, 17]. Mendoza et al. [11] summarized that it was suitable to use a constant mechanical loss to the open-drive expander without generator, whereas the proportional loss should be applied if an electric generator was connected to the expander since the generator loss was proportional to the generated power. Mendoza et al. obtained a nearly linear increase of torque loss between 0.1 N m to 0.6 N m as the rotational speed increases from 1536 rev min^{-1} to 3018 rev min^{-1}.

3.4 Motion equation of the generator

Similar to the rotation equation of a direct current (DC) motor suggested by [35-36], the following equation can be used to describe the rotation of a DC generator

$$T_{\mathrm{Drive}} = (J_{\mathrm{arm}} + J_{\mathrm{sha}}) \frac{\mathrm{d}\omega}{\mathrm{d}t} + B\omega + T_{\mathrm{em}} \tag{65}$$

where T_{Drive} is the torque transmitted from the shaft, B is the viscous friction coefficient. The above equation suggests that the mechanical loss can be expressed by the production of a friction coefficient and the rotational speed. Thus, the following equation can used to describe the friction torque in both scroll and generator [4-5] for the usage in Eq. (63)

$$T_{\mathrm{fr}} = f_{\mathrm{SG}}\omega \tag{66}$$

where f_{SG} is the overall dynamic friction coefficient of the scroll and generator system, and has the unit of N m s^{-1}. However, up to the authors' best knowledge, the value of this parameter has not been ever reported.

For a DC motor, the electromechanical torque T_{em} is generated when the conductors are placed in a magnetic field and the current flows through these conductors. T_{em} can be calculated by the following equation [35]

$$T_{\mathrm{em}} = K_{\mathrm{t}} I_{\mathrm{a}} \tag{67}$$

where K_{t} is the torque sensitivity, I_{a} is the charged current from the motor terminal. Eq. (67) can also be applied to a generator. On the other side, the generated current and the angular speed is coupled by another equation [35] as

$$E_{\mathrm{a}} = K_{\mathrm{e}}\omega = (R_{\mathrm{arm}} + R_{\mathrm{load}})I_{\mathrm{a}} + L_{\mathrm{a}}\frac{\mathrm{d}I_{\mathrm{a}}}{\mathrm{d}t} \tag{68}$$

where E_{a} is the back electromotive force (EMF), K_{e} is the constant of back EMF, L_{a} is the armature inductance, R_{a} and R_{load} are the resistances of the armature and electric load respectively. K_{e} and K_{t} have the same value if the SI unit is used [35]. Eqs. (66) to (68) are used to complete the motion equation of Eq. (63), however, f_{SG} must be determined previously.

For an AC generator, the modelling is more sophisticated, of which literature [6] can be referred.

3.5 Energy equation of the working fluid

The energy equation of the working fluid in the expansion chambers and discharge chamber is built according to the first law of thermodynamics

$$\frac{\mathrm{d}(mu)}{\mathrm{d}t} = \dot{Q} \quad [illegible] \tag{69}$$

where m is the mass in the chamber, u is the specific internal energy, $\dot{Q}$ is the rate of heat transfer into the chamber, $\dot{W}$ is the rate of work output of the expansion; $\dot{m}_{\mathrm{C\text{-}in}}$ and $\dot{m}_{\mathrm{C\text{-}out}}$ are the inlet and outlet mass flow rates of each chamber, while $h_{\mathrm{C\text{-}in}}$ and $h_{\mathrm{C\text{-}out}}$ are the corresponding fluid specific enthalpies.

Considering the work output equals to $P\mathrm{d}V/\mathrm{d}t$, the enthalpy $h = u + pV/m$, while the outlet enthalpy of the fluid h_{out} equals to the chamber enthalpy h_{C}, the above equation can be re-written as

$$\frac{\mathrm{d}(mh_{\mathrm{C}})}{\mathrm{d}t} = \dot{Q} \ldots \frac{\ldots}{\mathrm{d}t} \ldots \tag{70}$$

During the scroll rotation, the working fluid in each chamber exchange heat with the scrolls and base plates. In order to calculate the heat transfer rate, the heat transfer coefficient of the working fluid is necessary. Jang and Jeong [37] experimentally analogized the heat transfer associating with the scroll rotation to the heat transfer in a rectangular duct with a fixed heating wall and an oscillating wall. The corresponding empirical heat transfer correlation was then given as

$$Nu = (1 + 3.5\frac{D_{\mathrm{h}}}{D_{\mathrm{c}}})(1 + 8.8(1 - e^{-5.35St}))Nu_{\mathrm{DB}} \tag{71}$$

The term inside the first bracket on the right side of the correlation is the factor for the curved duct, D_{h} is the hydraulic diameter of the flow duct and D_{c} is the mean diameter of the scroll curvature, which can be calculated by the following equation for each chamber i at an orbiting angle θ

$$D_{\mathrm{c}} = a[(\pi/2 + \theta + 2(i-1)\pi) + (-\pi/2 + \theta + 2(i-1)\pi)] = 2a(\theta + 2(i-1)\pi) \tag{72}$$

The term in the second bracket on the right side of Eq. (71) is the correct factor for the oscillating movement, where St is the Strouhal number, a non-dimensional frequency factor and is given by

$$St = \frac{f_{\mathrm{o}} A_{\mathrm{max}}}{\bar{U}} \tag{73}$$

where f_{o} is the oscillating frequency, A_{max} is the oscillating amplitude, $\bar{U}$ is the mean flow velocity of the working fluid. For the scroll expander, these two factors can be given by

$$f_{\mathrm{o}} = \frac{\omega}{2\pi} \tag{74}$$

$$A_{\mathrm{max}} = R_{\mathrm{or}} \tag{75}$$

Here, A_{max} is set at R_{or} since $2R_{\mathrm{or}}$ is the maximum flow duct width in the scroll expander. Thus the flow duct in the scroll chamber is analogized to a rectangular duct with varying width from 0 to $2R_{\mathrm{or}}$ and scroll height h_{s}. Therefore, the mean hydraulic diameter and mean flow velocity can be given by

$$D_{\mathrm{h}}=\frac{2R_{\mathrm{or}}h_{\mathrm{s}}}{R_{\mathrm{or}}+h_{\mathrm{s}}} \tag{76}$$

$$\overline{U}=\frac{\dot{m}_{\mathrm{total}}}{R_{\mathrm{or}}h\rho} \tag{77}$$

Nu_{DB} in Eq. (71) is the Dittus-Boelter heat convection correlation, given as

$$Nu_{\mathrm{DB}}=0.023Re^{0.8}Pr^{1/3} \tag{78}$$

By neglecting the heat capacity of the scroll wraps and base plates, the heat transfer rate into the working fluid in chamber i is

$$\begin{aligned}\frac{\mathrm{d}Q_i}{\mathrm{d}t}=&\frac{T_{\mathrm{C}i+1}-T_{\mathrm{C}i}}{\frac{t}{\lambda_{\mathrm{scroll}}}+\frac{D_{\mathrm{h}}}{Nu_{\mathrm{C}i+1}\lambda_{\mathrm{WF}}}+\frac{D_{\mathrm{h}}}{Nu_{\mathrm{C}i}\lambda_{\mathrm{WF}}}}A_i+\frac{T_{\mathrm{C}i-1}-T_{\mathrm{C}i}}{\frac{t}{\lambda_{\mathrm{scroll}}}+\frac{D_{\mathrm{h}}}{Nu_{\mathrm{C}i-1}\lambda_{\mathrm{WF}}}+\frac{D_{\mathrm{h}}}{Nu_{\mathrm{C}i}\lambda_{\mathrm{WF}}}}A_{i-1}\\&+\frac{T_{\mathrm{amb}}-T_{\mathrm{C}i}}{\frac{t_{\mathrm{Plate}}}{\lambda_{\mathrm{Plate}}}+\frac{1}{\alpha_{\mathrm{amb}}}+\frac{D_{\mathrm{h}}}{Nu_{\mathrm{C}i}\lambda_{\mathrm{WF}}}}A_{\mathrm{p}i}\end{aligned} \tag{79}$$

where $\lambda_{\mathrm{scroll}}$ and λ_{WF} are the thermal conductivities of the scroll material and working fluid respectively, λ_{Plate} is the thermal conductivity of the base plate material; t_{s} and t_{Plate} are the thicknesses of the scroll and scroll base plate respectively; α_{amb} is the ambient natural convection heat transfer coefficient. For chamber 1, the second term on the right hand side of the above equation disappears; for the discharge chamber, the down-stream temperature $T_{\mathrm{C}i+1}$ is then the ambient temperature T_{amb}, while the heat transfer coefficient term $D_{\mathrm{h}}/Nu_{\mathrm{C}i+1}\lambda_{\mathrm{WF}}$ in the denominator changes to $1/\alpha_{\mathrm{amb}}$. The heat transfer area between chambers is

$$\begin{cases}A_i=L_{\mathrm{r}}h=2\pi ah\big(2(i-1)\pi+\theta\big) & i\ \le\ N_{\mathrm{C}}\\ A_i=2\pi R_{\mathrm{sh}}h & i=N_{\mathrm{C}}+1\end{cases} \tag{80}$$

The heat transfer area between chamber and scroll base plate is

$$A_{\mathrm{P}i}=\frac{2V_{\mathrm{C}i}}{h} \tag{81}$$

3.6 Scroll expander power and efficiency

Based on the thermodynamics, the total input working fluid energy can be calculated by

$$P_{\mathrm{WF}} = \dot{m} \quad {}_{\mathrm{in}} \; {}_{\mathrm{in}} \qquad \dot{} \quad {}_{\mathrm{out}} \; {}_{\mathrm{out}} \tag{82}$$

where $\dot{m}_{\mathrm{in}}$ and $\dot{m}_{\mathrm{out}}$ are the inlet and outlet mass flow rates to the scroll expander, h_{in} and h_{out} are the corresponding fluid specific enthalpies. The shaft power, P_{sha}, which stands for the power output from the crank shaft to drive the generator is calculated by

$$P_{\mathrm{sha}} = T_{\mathrm{Drive}}\omega \tag{83}$$

The scroll efficiency and the overall system efficiency are defined as

$$\eta_{\mathrm{scroll}} = \frac{P_{\mathrm{sha}}}{P_{\mathrm{WF}}} \tag{84}$$

$$\eta_{\mathrm{overall}} = \frac{P_{\mathrm{load}}}{P_{\mathrm{WF}}} \tag{85}$$

where P_{load} is the power consumed by the electric load.